Functional central limit theorems and moderate deviations for Poisson cluster processes

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This talk is based on joint works with Yujing Wang

Outline

Poisson cluster process

- A Poisson cluster process $\mathbf{X} \subset \mathbb{R}$ is a point process generated from an immigrant process and a family of offspring processes.
  - The immigrant process I is a homogeneous Poisson process with points $X_i \in \mathbb{R}$ and intensity $\nu > 0$.
  - Each immigrant X_i generates a cluster, i.e., offspring process $C_i = C_{X_i}$ which is a finite point process.
  - Given the immigrants, the centered clusters

$$C_i - X_i = \{Y - X_i : Y \in C_i\}, \quad X_i \in I$$

    are independent, identically distributed and independent of I.
  - $\mathbf{X}$ consists of the union of all clusters.

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- Let S denote the number of points in a cluster.
- Let $N_{\mathbf{X}}(0,t]$ denote the number of points of $\mathbf{X}$ in the interval $(0,t]$.
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Hawkes process

- Linear Hawkes process is a class of Poisson cluster processes. Its each cluster $C_i = C_{X_i}$ has the following branching structure:
  - The immigrant X_i is said to be of generation 0.
  - Given generations $0, 1, \cdots, n$ in C_i, each point $Y \in C_i$ of generation n generates a Poisson process on (Y, ∞) of offspring of generation $n+1$ with intensity function $h(\cdot - Y)$, where $h : (0, \infty) \to [0, \infty)$ is a non-negative Borel function.
- Hawkes process can be represented by a SDE

$$Z_t := \int_0^t \int_0^\infty 1_{[0, \phi(\int_0^{s-} h(s-u) dZ_u^\varepsilon)]}(z) \pi(dzds). \quad (1.1)$$

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- As usual, we assume that

$$\mu := \int_0^\infty h(t)dt \in (0,1) \qquad (B1)$$

and

$$\int_0^\infty th(t)dt < \infty. \qquad (B2)$$

- It is known that

$$P(S=k) = \frac{e^{-k\mu}(k\mu)^{k-1}}{k!}, \quad k = 1, 2, \cdots \qquad (1.2)$$

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Background

- Poisson cluster processes are an important class of point process models (see Daley and Vere-Jones (2003)).
- The linear Hawkes process was first proposed by A. Hawkes to model earthquakes and their aftershocks (**Biometrika**, 1971)
- The nonlinear Hawkes process was first introduced by Brémaud and Massoulié (**Ann. Probab.**, 1996).

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Background

- Long time behaviors of Hawkes processes have been studied widely, for examples,
  - **FCLT:** see Bacry et al. (linear case, SPA,2013), Zhu (nonlinear case, JAP (2013))
  - **LDP:** see Bordenave and Torrisi (linear case, 2007), Zhu (nonlinear case, AIHP (2014), AOAP (2015).
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- Small perturbation problems for the nonlinear Hawkes process

$$Z_t^{\varepsilon} := \varepsilon \int_0^t \int_0^{\infty} 1_{[0, \frac{1}{\varepsilon}\phi(\int_0^{s-} h(s-u)dZ_u^{\varepsilon})]}(z)\pi(dzds). \qquad (1.3)$$

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- The mean fields of Hawkes processes:

$$Z_t^{N,i} = \int_0^t \int_0^\infty I_{\left\{z \le \phi\left(N^{-1}\sum_{j=1}^N \int_0^{s-} h(s-u)dZ_u^{N,j}\right)\right\}} \pi^i(dz\,ds).$$

The *mean field* is defined by

$$L^N(t,dx) = \frac{1}{N}\sum_{i=1}^N \delta_{Z_t^{N,i}}(dx), \quad 0 \le t \le T.$$

- **LLN and Fluctuations:** see Delattre, Fournier, and Hoffmann (AOAP, 2016), Chevallier (SPA, 2017).
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- Our motivation: Consider functional central limit theorem and functional moderate deviation principle for Poisson cluster processes.
- Limit in $(D[0,1],\rho_s)$ for $\frac{N_{\mathbf{X}}(0,\alpha t]-E(N_{\mathbf{X}}(0,\alpha t])}{\sqrt{\alpha}}$, $t\in[0,1]$.
- LDP in $(D[0,1],\rho_s)$ for $\frac{N_{\mathbf{X}}(0,\alpha t]-E(N_{\mathbf{X}}(0,\alpha t])}{b(\alpha)}$, $t\in[0,1]$, where $\{b(\alpha),\alpha>0\}$ is a positive function satisfying:

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FCLT

Theorem 2.1

Assume that (A1) holds. Then as $\alpha \to \infty$,

$$\frac{N_{\mathbf{X}}(0, \alpha t] - E(N_{\mathbf{X}}(0, \alpha t])}{\sqrt{\alpha}} \xrightarrow{d} \sigma B(t)$$

in $(D[0,1], \rho_s)$, where $\{B(t), t \geq 0\}$ is the standard Brownian motion, and $\xrightarrow{d}$ denotes convergence in distribution.

Theorem 2.2

Assume that (A2) holds. Define $J : D[0,1] \to [0,\infty]$ *as follows*

$$J(f) = \begin{cases} \dfrac{1}{2\sigma^2}\displaystyle\int_0^1 |\dot{f}(t)|^2 dt, & \text{if } f \in AC_0[0,1]; \\ +\infty, & \text{otherwise.} \end{cases} \quad (2.1)$$

Then $\left\{ \frac{N_{\mathbf{X}}(0,\alpha t] - E(N_{\mathbf{X}}(0,\alpha t])}{b(\alpha)}, t \in [0,1] \right\}$ *satisfies the large deviation principle (LDP) on* $(D[0,1], \|\cdot\|)$ *with speed* $\frac{b^2(\alpha)}{\alpha}$ *and good rate function* $J(f)$.

That is,
(1). For any $l \leq 0$, $\{f; J(f) \leq l\}$ is compact in $(D[0,1], \|\cdot\|)$;
(2). For any closed F in $(D[0,1], \|\cdot\|)$,

$$\limsup_{\alpha\to\infty} \frac{\alpha}{b^2(\alpha)} \log P\left(\frac{N_{\mathbf{X}}(0,\alpha\cdot] - E(N_{\mathbf{X}}(0,\alpha\cdot])}{b(\alpha)} \in F\right) \leq -\inf_{f\in F} J(f),$$

and for any open G in $(D[0,1], \|\cdot\|)$,

$$\liminf_{\alpha\to\infty} \frac{\alpha}{b^2(\alpha)} \log P\left(\frac{N_{\mathbf{X}}(0,\alpha\cdot] - E(N_{\mathbf{X}}(0,\alpha\cdot])}{b(\alpha)} \in G\right) \geq -\inf_{f\in G} J(f).$$

Corollary 2.1

Let X be a Hawkes process. Assume that (B1) and (B2) hold. Then

(1). $\left\{\frac{N_{\mathbf{X}}(0,\alpha t]-E(N_{\mathbf{X}}(0,\alpha t])}{\sqrt{\alpha}}, t \in [0,1]\right\}$ *converges in distribution to* $\sqrt{\frac{\nu}{(1-\mu)^3}}B(t)$ *in* $(D[0,1], \rho_s)$.

(2). $\left\{\frac{N_{\mathbf{X}}(0,\alpha t]-E(N_{\mathbf{X}}(0,\alpha t])}{b(\alpha)},\ t \in [0,1]\right\}$ *satisfies the large deviation principle (LDP) on* $(D[0,1], \|\cdot\|)$ *with speed* $\frac{b^2(\alpha)}{\alpha}$ *and good rate function* $J^H(f)$ *defined by*

$$J^H(f) = \begin{cases} \frac{(1-\mu)^3}{2\nu}\int_0^1 |\dot{f}(t)|^2 dt, & \text{if } f \in AC_0[0,1]; \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.2}$$

A maximal inequality for Poisson cluster processes

Define

$$C(t) = \sum_{X_k \in I_{|(0,t]}} N_{C_k}(0, t]. \tag{2.3}$$

We present a maximal inequality for Poisson cluster processes. It plays a very important role in this paper.

A maximal inequality for Poisson cluster processes

Lemma 2.1

Let $0 \le s < t$*, and* $s = t_0 < t_1 < \cdots < t_n = t$*. Then for any* $r > 0$*,*

$$\begin{aligned} & P\left(\max_{1\le l\le n} |C(t_l) - C(s) - E(C(t_l) - C(s))| > 3r\right) \\ \le & 2P\left(\max_{0\le l\le n-1}\left|\sum_{X_k \in I_{|(0,t_l]}} N_{C_k}(t_l, t_n] - E\left(\sum_{X_k \in I_{|(0,t_l]}} N_{C_k}(t_l, t_n]\right)\right| > r/2\right) \\ & + 2\max_{0\le l\le n-1} P\left(\left|\sum_{X_k \in I_{|(t_l,t_n]}} N_{C_k}(0, t_n] - E\left(\sum_{X_k \in I_{|(t_l,t_n]}} N_{C_k}(0, t_n]\right)\right| > r/2\right). \end{aligned} \tag{2.4}$$

Proof of FCLT

In this section, we give a proof of the following FCLT.

Theorem 3.1

Assume that (A1) holds. Then as $\alpha \to \infty$,

$$\frac{N_{\mathbf{X}}(0, \alpha t] - E(N_{\mathbf{X}}(0, \alpha t])}{\sqrt{\alpha}} \xrightarrow{d} \sigma B(t)$$

in $(D[0,1], \rho_s)$, *where* $\{B(t), t \geq 0\}$ *is the standard Brownian motion, and* $\xrightarrow{d}$ *denotes convergence in distribution.*

Proof of FCLT

Let us write

$$\frac{N_{\mathbf{X}}(0,\alpha t] - E(N_{\mathbf{X}}(0,\alpha t]))}{\sqrt{\alpha}}$$

$$= \frac{C(\alpha t) - E(C(\alpha t))}{\sqrt{\alpha}}$$

$$+ \frac{\sum_{X_k \in I_{|(-\infty,0]}} N_{C_k}(0,\alpha t] - E(\sum_{X_k \in I_{|(-\infty,0]}} N_{C_k}(0,\alpha t])}{\sqrt{\alpha}}$$

$$+ \frac{\sum_{X_k \in I_{|(\alpha t,\infty]}} N_{C_k}(0,\alpha t] - E(\sum_{X_k \in I_{|(\alpha t,\infty]}} N_{C_k}(0,\alpha t]}{\sqrt{\alpha}}.$$

We will prove that the second term and the third term are negligible and

$$\frac{C(\alpha t) - E(C(\alpha t))}{\sqrt{\alpha}} \xrightarrow{d} \sigma B(t).$$

Proof of FCLT

That is,

(a). $\frac{1}{\sqrt{\alpha}} \sup_{t\in[0,1]} \left| \sum_{X_k \in I_{|(-\infty,0]}} N_{C_k}(0,\alpha t] - E\left(\sum_{X_k \in I_{|(-\infty,0]}} N_{C_k}(0,\alpha t] \right) \right| \to 0$ in probability

(b). $\frac{1}{\sqrt{\alpha}} \sup_{t\in[0,1]} \left| \sum_{X_k \in I_{|(\alpha t,\infty)}} N_{C_k}(0,\alpha t] - E\left(\sum_{X_k \in I_{|(\alpha t,\infty)}} N_{C_k}(0,\alpha t] \right) \right| \to 0$ in probability

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Next let us show the following result.

Proposition 3.1

Assume that (A1) holds. Then

$$\frac{C(\alpha t) - E(C(\alpha t))}{\sqrt{\alpha}} \xrightarrow{d} \sigma B(t) \tag{3.1}$$

in $(D[0,1], \rho_s)$.

It is sufficient to show the following two lemmas.

Lemma 3.1

Assume that (A1) holds. Then for each $n \geq 1$ *and* $0 \leq t_1 < \cdots < t_n \leq 1$,

$$\left(\frac{C(\alpha t_1) - E(C(\alpha t_1))}{\sqrt{\alpha}}, \cdots, \frac{C(\alpha t_n) - E(C(\alpha t_n))}{\sqrt{\alpha}}\right) \xrightarrow{d} \sigma(B(t_1), \cdots, B(t_n)).$$

Lemma 3.2

Assume that (A1) satisfies. Then for any $\delta > 0$,

$$\lim_{\eta \to 0} \limsup_{\alpha \to \infty} P\left(\sup_{|t-s|<\eta} \Big| C(\alpha t) - E(C(\alpha t)) - (C(\alpha s) - E(C(\alpha s)))\Big| > 3\sqrt{\alpha}\delta\right) = 0. \tag{3.2}$$

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Lemma 3.2

Assume that (A1) satisfies. Then for any $\delta > 0$,

$$\lim_{\eta \to 0} \limsup_{\alpha \to \infty} P\Bigg(\sup_{|t-s|<\eta} \Big| C(\alpha t) - E(C(\alpha t)) - (C(\alpha s) - E(C(\alpha s))) \Big| > 3\sqrt{\alpha}\delta \Bigg) = 0. \tag{3.2}$$

Proof of Lemma 3.1

By the definition of Poisson cluster process, we have that

$$E\left(e^{\frac{\sqrt{-1}}{\sqrt{\alpha}}\sum_{i=1}^{n}\theta_i\left(\sum_{X_k\in I_{|(0,\alpha t_i]}}N_{C_k}(0,\alpha t_i]-E\left(\sum_{X_k\in I_{|(0,\alpha t_i]}}N_{C_k}(0,\alpha t_i]\right)\right)}\right)$$

$$=\exp\left\{\nu\sum_{j=1}^{n}\int_0^{\alpha(t_j-t_{j-1})}\left\{E\left(e^{\frac{\sqrt{-1}}{\sqrt{\alpha}}\sum_{i=j}^{n}\theta_i N_{C_0}(-\alpha t_{j-1}-s,\alpha(t_i-t_{j-1})-s]}-1\right)\right.\right.$$

$$\left.\left.-\frac{\sqrt{-1}}{\sqrt{\alpha}}E\left(\sum_{i=j}^{n}\theta_i N_{C_0}(-\alpha t_{j-1}-s,\alpha(t_i-t_{j-1})-s]\right)\right\}ds\right\}$$

Note that $N_{C_0}(-\alpha t_{j-1}-s, \alpha(t_i-t_{j-1})-s] \le N_{C_0}(\mathbb{R}) = S$ and $N_{C_0}(-\alpha t_{j-1}-s, \alpha(t_i - t_{j-1}) - s] \uparrow S$ as $\alpha \uparrow$. Thus

$$\lim_{\alpha\to\infty} E\left(e^{\frac{\sqrt{-1}}{\sqrt{\alpha}}\sum_{i=1}^n \theta_i\left(\sum_{X_k\in I_{|(0,\alpha t_i]}} N_{C_k}(0,\alpha t_i]-E\left(\sum_{X_k\in I_{|(0,\alpha t_i]}} N_{C_k}(0,\alpha t_i]\right)\right)}\right)$$

$$=\exp\left\{-\frac{1}{2}\nu E(S^2)\sum_{j=1}^n\left(\sum_{i=j}^n \theta_i\right)^2 (t_j - t_{j-1})\right\}$$

$$=E\left(e^{\sqrt{-1}\sum_{i=1}^n \theta_i \sigma B(t_i)}\right).$$

Proof of Lemma 3.2

For $\eta > 0$ given, set

$$A_s(\delta) = \left\{ \sup_{s \le t \le s+\eta} \left| C(\alpha t) - E(C(\alpha t)) - (C(\alpha s) - E(C(\alpha s))) \right| > \sqrt{\alpha}\delta \right\}.$$

Then

$$P\left(\sup_{|t-s|<\eta} \left| C(\alpha t) - E(C(\alpha t)) - (C(\alpha s) - E(C(\alpha s))) \right| > 3\sqrt{\alpha}\delta \right)$$
$$\le P\left(\cup_{i \le \eta^{-1}} A_{i\eta}(\delta)\right) \le \left(1 + \frac{1}{\eta}\right) \sup_{s\in(0,1)} P\left(A_s(\delta)\right)$$

Thus, if for any $\delta > 0$,

$$\lim_{\eta\to 0} \limsup_{\alpha\to\infty} \frac{1}{\eta} \sup_{s\in(0,1)} P\left(A_s(\delta)\right) = 0, \tag{3.3}$$

then (3.2) holds. Thus, we only need to prove (3.3).

For any $\eta \in (0,1)$, $s \in (0,1)$, set $t_l = s\alpha + l\eta$ for $l = 0, 1, \cdots, [\alpha]+1$. Then for any $\delta > 0$,

$$
\begin{aligned}
&P\left(\sup_{s\le t\le s+\eta}\left|C(\alpha t) - E(C(\alpha t)) - (C(\alpha s) - E(C(\alpha s)))\right| > \sqrt{\alpha}\delta\right)\\
\le &P\left(\max_{1\le l\le[\alpha]+1}\left|C(t_l) - E(C(t_l)) - (C(\alpha s) - E(C(\alpha s)))\right| > \sqrt{\alpha}\delta/2\right)\\
&+ P\left(\max_{0\le l\le[\alpha]}\left|E(C(t_l) - C(t_{l+1}))\right| > \sqrt{\alpha}\delta/2\right).
\end{aligned}
$$

By the maximal inequality for poisson cluster processes (lemma 2.1),

$$P\left(\max_{1\le l\le[\alpha]+1}\left|C(t_l)-E(C(t_l))-(C(\alpha s)-E(C(\alpha s)))\right|>\sqrt{\alpha}\delta/2\right)$$

$$\le 2P\left(\max_{0\le l\le[\alpha]}\left|\sum_{X_k\in I_{|(0,t_l]}}N_{C_k}(t_l,t_{[\alpha]+1}]-E\left(\sum_{X_k\in I_{|(0,t_l]}}N_{C_k}(t_l,t_{[\alpha]+1}]\right)\right|>\frac{\sqrt{\alpha}\delta}{12}\right)$$

$$+2\max_{0\le l\le[\alpha]}P\left(\left|\sum_{X_k\in I_{|(t_l,t_{[\alpha]+1}]}}N_{C_k}(0,t_{[\alpha]+1}]\right.\right.$$

$$\left.\left.-E\left(\sum_{X_k\in I_{|(t_l,t_{[\alpha]+1}]}}N_{C_k}(0,t_{[\alpha]+1}]\right)\right|>\frac{\sqrt{\alpha}\delta}{12}\right).$$

Thus, we only need to show that

$$\sup_{s\in(0,1)} P\left(\max_{0\le l\le[\alpha]} E(C(t_{l+1})-C(t_l)) > \sqrt{\alpha}\delta/2\right) \to 0, \tag{3.4}$$

$$\begin{aligned}\sup_{s\in(0,1)} P\Bigg(\max_{0\le l\le[\alpha]}\Bigg| &\sum_{X_k\in I_{|(0,t_l]}} N_{C_k}(t_l, t_{[\alpha]+1}] \\ &- E\Bigg(\sum_{X_k\in I_{|(0,t_l]}} N_{C_k}(t_l, t_{[\alpha]+1}]\Bigg)\Bigg| > \frac{\sqrt{\alpha}\delta}{12}\Bigg) \to 0.\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\frac{1}{\eta}\sup_{s\in(0,1)}\max_{0\le l\le[\alpha]} P\Bigg(\Bigg| &\sum_{X_k\in I_{|(t_l,t_{[\alpha]+1}]}} N_{C_k}(0, t_{[\alpha]+1}] \\ &- E\Bigg(\sum_{X_k\in I_{|(t_l,t_{[\alpha]+1}]}} N_{C_k}(0, t_{[\alpha]+1}]\Bigg)\Bigg| > \sqrt{\alpha}\delta/12\Bigg) \to 0.\end{aligned} \tag{3.6}$$

Notice that $\sum_{X_k \in I_{|(-\infty,t_l]}} N_{C_k}(t_l,\infty)$, $l = 0, 1, \cdots, [\alpha]+1$ are identically distributed with $\sum_{X_k \in I_{|(-\infty,0]}} N_{C_k}(0,\infty)$, and

$$E\left(\sum_{X_k \in I_{|(-\infty,t_l]}} N_{C_k}(t_l,\infty)\right) \leq \nu E(LS),$$

$$E\left(\left(\sum_{X_k \in I_{|(-\infty,t_l]}} N_{C_k}(t_l,\infty)\right)^2\right) \leq (\nu E(LS))^2 + \nu E(LS^2).$$

Therefore, for α large enough,

$$
\begin{aligned}
&P\left(\max_{0\le l\le[\alpha]}\left|\sum_{X_k\in I_{|(0,t_l]}} N_{C_k}(t_l,t_{[\alpha]+1}] - E\left(\sum_{X_k\in I_{|(0,t_l]}} N_{C_k}(t_l,t_{[\alpha]+1}]\right)\right| > \frac{\sqrt{\alpha}\delta}{12}\right)\\
\le&(\alpha+1)\max_{0\le l\le[\alpha]} P\left(\sum_{X_k\in I_{|(0,t_l]}} N_{C_k}(t_l,t_{[\alpha]+1}] > \frac{\sqrt{\alpha}\delta}{20}\right)\\
\le&(\alpha+1)P\left(\sum_{X_k\in I_{|(-\infty,0]}} N_{C_k}(0,\infty) > \sqrt{\alpha}\delta/20\right)
\end{aligned}
$$

Now, by Chebyshev's inequality, we can get (3.4),(3.5) and (3.6).

Proof of FMDP

In this section, we give a proof of the following moderate deviation principle.

Theorem 4.1

Assume that (A2) holds. Define $J : D[0,1] \to [0,\infty]$ *as follows*

$$J(f) = \begin{cases} \frac{1}{2\sigma^2}\int_0^1 |\dot{f}(t)|^2 dt, & \text{if } f \in AC_0[0,1]; \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.1}$$

Then $\left\{ \frac{N_{\mathbf{X}}(0,\alpha t] - E(N_{\mathbf{X}}(0,\alpha t])}{b(\alpha)}, t \in [0,1] \right\}$ *satisfies the large deviation principle (LDP) on* $(D[0,1], \|\cdot\|)$ *with speed* $\frac{b^2(\alpha)}{\alpha}$ *and good rate function* $J(f)$.

Proof of FMDP

It is obvious that

$$\begin{aligned}
&\frac{N_{\mathbf{X}}(0,\alpha t] - E(N_{\mathbf{X}}(0,\alpha t])}{b(\alpha)}\\
=&\frac{C(\alpha t) - E(C(\alpha t))}{b(\alpha)}\\
&+\frac{\sum_{X_k \in I_{|(-\infty,0]}} N_{C_k}(0,\alpha t] - E(\sum_{X_k \in I_{|(-\infty,0]}} N_{C_k}(0,\alpha t])}{b(\alpha)}\\
&+\frac{\sum_{X_k \in I_{|(\alpha t,\infty]}} N_{C_k}(0,\alpha t] - E(\sum_{X_k \in I_{|(\alpha t,\infty]}} N_{C_k}(0,\alpha t]}{b(\alpha)}.
\end{aligned}$$

We will show that

(a). $\frac{N_{\mathbf{X}}(0,\alpha t] - E(N_{\mathbf{X}}(0,\alpha t])}{b(\alpha)}$ and $\frac{C(\alpha t) - E(C(\alpha t))}{b(\alpha)}$ are exponentially equivalent in moderate deviation.

(b). $\frac{C(\alpha t) - E(C(\alpha t))}{b(\alpha)}$ satisfies MDP.

Proof of FMDP

It is obvious that

$$\begin{aligned}
&\frac{N_{\mathbf{X}}(0,\alpha t]-E(N_{\mathbf{X}}(0,\alpha t])}{b(\alpha)} \\
=&\frac{C(\alpha t)-E(C(\alpha t))}{b(\alpha)} \\
&+\frac{\sum_{X_k\in I_{|(-\infty,0]}} N_{C_k}(0,\alpha t]-E(\sum_{X_k\in I_{|(-\infty,0]}} N_{C_k}(0,\alpha t])}{b(\alpha)} \\
&+\frac{\sum_{X_k\in I_{|(\alpha t,\infty]}} N_{C_k}(0,\alpha t]-E(\sum_{X_k\in I_{|(\alpha t,\infty]}} N_{C_k}(0,\alpha t]}{b(\alpha)}.
\end{aligned}$$

We will show that

(a). $\frac{N_{\mathbf{X}}(0,\alpha t]-E(N_{\mathbf{X}}(0,\alpha t])}{b(\alpha)}$ and $\frac{C(\alpha t)-E(C(\alpha t))}{b(\alpha)}$ are exponentially equivalent in moderate deviation.

(b). $\frac{C(\alpha t)-E(C(\alpha t))}{b(\alpha)}$ satisfies MDP.

Exponential equivalence:
For any $\delta > 0$,

$$\lim_{\alpha\to\infty} \frac{\alpha}{b^2(\alpha)} \log P\left(\sup_{t\in[0,1]} \left| \sum_{X_k \in I_{|(-\infty,0]}} N_{C_k}(0,\alpha t] - E\left(\sum_{X_k \in I_{|(-\infty,0]}} N_{C_k}(0,\alpha t] \right) \right| > b(\alpha)\delta \right) = -\infty, \tag{4.2}$$

and

$$\lim_{\alpha\to\infty} \frac{\alpha}{b^2(\alpha)} \log P\left(\sup_{t\in[0,1]} \left| \sum_{X_k \in I_{|(\alpha t,\infty)}} N_{C_k}(0,\alpha t] - E\left(\sum_{X_k \in I_{|(\alpha t,\infty)}} N_{C_k}(0,\alpha t] \right) \right| > b(\alpha)\delta \right) = -\infty. \tag{4.3}$$

Proposition 4.1

Assume that (A2) holds. Then $\left\{\frac{C(\alpha t)-E(C(\alpha t))}{b(\alpha)},\ t\in[0,1]\right\}$ *satisfies the large deviation principle (LDP) on* $(D[0,1],\|\cdot\|)$ *with speed* $\frac{b^2(\alpha)}{\alpha}$ *and good rate function* $J(f)$.

It is sufficient to show the following two Lemmas.

Lemma 4.1

Assume that (A2) holds. Then for each $n \geq 1$ and $0 \leq t_1 < \cdots < t_n \leq 1$,

$$\left(\frac{C(\alpha t_1) - E(C(\alpha t_1))}{b(\alpha)}, \cdots, \frac{C(\alpha t_n) - E(C(\alpha t_n))}{b(\alpha)}\right)$$

satisfies the LDP in $\mathbb{R}^n$ with speed $\frac{b^2(\alpha)}{\alpha}$ and good rate function $J_{t_1,\cdots,t_n}(x_1, \cdots, x_n)$.

Lemma 4.2

Assume that (A2) satisfies. Then for any $\delta > 0$, $s \in (0, 1)$,

$$\lim_{\eta\to 0} \limsup_{\alpha\to\infty} \frac{\alpha}{b^2(\alpha)} \log P\Big(\sup_{s\le t\le s+\eta} \Big| C(\alpha t) - E(C(\alpha t)) - (C(\alpha s) - E(C(\alpha s))) \Big| > b(\alpha)\delta \Big) = -\infty. \quad (4.4)$$

Thank You !

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